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Distributions of Orientations on Stiefel Manifolds

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The Riemann space whose elements are $m \times k$ $(m \geqq k)$ matrices X, i.e., orientations, such that $X'X = I_k$ is called the Stiefel manifold $V_{k,m}$. The matrix Langevin (or von Mises–Fisher) and matrix Bingham distributions have been suggested as distributions on $V_{k,m}$. In this paper, we present some distributional results on $V_{k,m}$. Two kinds of decomposition are given of the differential form for the invariant measure on $V_{k,m}$, and they are utilized to derive distributions on the component Stiefel manifolds and subspaces of $V_{k,m}$ for the above-mentioned two distributions. The singular value decomposition of the sum of a random sample from the matrix Langevin distribution gives the maximum likelihood estimators of the population orientations and modal orientation. We derive sampling distributions of matrix statistics including these sample estimators. Furthermore, representations in terms of the Hankel transform and multi-sample distribution theory are briefly discussed.

1. INTRODUCTION

An orientation is defined as a rigid k-frame in m dimensions ($k \leq m$), i.e., an $m \times k$ matix X such that $X'X = C$, where C is a $k \times k$ positive definite matrix specifying the angles between the columns of X (see Downs [7]). Without loss of generality we suppose $C = I_k$, the $k \times k$ identity matrix, since all methodology for $C = I_k$ can be extended for general C. The Riemann space whose elements are $m \times k$ matrices X such that $X'X = I_k$ is called the Stiefel manifold and denoted by $V_{k,m}$. For $k = m$, the Stiefel manifold is the orthogonal group $O(m)$. Practical examples of data on $V_{k,m}$ are illustrated by Downs [7] in vector cardiography and by Jupp and Mardia [12] in astronomy.

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An invariant measure on $V_{k,m}$ is given by the differential form

$$(X'\,dX) \equiv \bigwedge_{i<j}{}_1^k\, \mathbf{x}_j'\,d\mathbf{x}_i \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k} \mathbf{x}_{k+j}'\,d\mathbf{x}_i, \tag{1.1}$$

in terms of the exterior products ($\bigwedge$), where we choose an $m \times (m-k)$ matrix X_1 such that $(X : X_1) = (\mathbf{x}_1 \cdots \mathbf{x}_k : \mathbf{x}_{k+1} \cdots \mathbf{x}_m) \in O(m)$ and $d\mathbf{x}$ is an $m \times 1$ vector of differentials. See, e.g., Muirhead [15] for the use of exterior products. The volume of $V_{k,m}$ is

$$w(k, m) = \int_{V_{k,m}} (X'\,dX) = 2^k \pi^{km/2} / \Gamma_k(m/2), \tag{1.2}$$

where $\Gamma_k(a) = \pi^{k(k-1)/4} \prod_{i=1}^k \Gamma(a-(i-1)/2)$. We denote the normalized invariant measure of unit mass on $V_{k,m}$ by $[dX]$ ($\equiv (X'\,dX)/w(k, m)$). See James [10] and Farrell [8, Chaps. 6–8] for detailed discussion of manifolds and their invariant measures.

The matrix Langevin (or von Mises–Fisher) distribution, denoted by $L(m, k; F)$, was defined by Downs [7] to have density

$$[{}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}F'F)]^{-1}\, etr(F'X), \tag{1.3}$$

with respect to $[dX]$, where F is an $m \times k$ matrix, and the ${}_0F_1^{(k)}$ is a hypergeometric function of matrix argument. The general hypergeometric function ${}_pF_q^{(r)}(a_1, ..., a_p; b_1, ..., b_q; S)$ of a $k \times k$ symmetric matrix S ($r \leqq k$) has a representation in terms of zonal polynomials

$$\sum_{l=0}^{\infty} \sum_{\lambda} [(a_1)_\lambda \cdots (a_p)_\lambda / (b_1)_\lambda \cdots (b_q)_\lambda\, l!]\, C_\lambda(S),$$

where $\lambda = (l_1, ..., l_r)$, $l_1 \geqq \cdots \geqq l_r > 0$, $\sum_{i=1}^r l_i = l$, $(a)_\lambda = \prod_{i=1}^r (a-(i-1)/2)_{l_i}$, $(a)_l = a(a+1) \cdots (a+l-1)$, and $C_\lambda(S)$ is a zonal polynomial. $C_\lambda(S)$ is a homogeneous symmetric polynomial of degree l in the latent roots of S. See James [11] and Constantine [5] for detailed discussion of zonal polynomials and hypergeometric functions of matrix arguments. The $L(m, k; F)$ distribution is a uni-modal and rotationally symmetric distribution. The component matrices in the singular value decomposition of F can be interpreted as orientations and concentrations. Distribution theory, maximum likelihood estimates, and likelihood ratio tests have been developed by Downs [7], Jupp and Mardia [12], and Khatri and Mardia [13].

It is noted here that, throughout this paper, densities of distributions of a random matrix X on the Stiefel manifold are expressed with respect to the normalized measure $[dX]$, while densities of distributions of any $q \times k$

random matrix $Y=(y_{ij})$ are expressed with respect to the measure (dY). Here we denote

$$(dY) \equiv \bigwedge_{j=1}^{k} \bigwedge_{i=1}^{q} dy_{ij}$$

$$\equiv \bigwedge_{1 \leq i \leq j \leq k} dy_{ij}, \qquad \text{if} \quad Y \text{ is } k \times k \text{ symmetric.}$$

The matrix Bingham distribution, denoted by $B(m, k; A)$, has density

$$[{}_1F_1^{(k)}(\tfrac{1}{2}k; \tfrac{1}{2}m; A)]^{-1} \operatorname{etr}(X'AX), \tag{1.4}$$

where A is an $m \times m$ symmetric matrix with a restriction imposed to ensure the identifiability of A, e.g., tr $A=0$. This distribution is an extension of the Bingham distribution (Bingham [1]) on the sphere and a particular case of a distribution introduced by Khatri and Mardia [13] and has been furthermore generalized by Prentice [16]. Jupp and Mardia [12] and Prentice [16] discuss statistical inferences on the matrix Bingham distribution. The case $F=0$ or $A=0$ in (1.3) or (1.4), respectively, reduces to the uniform distribution $[dX]$ on $V_{k,m}$. It is noted that the matrix angular central Gaussian distribution is proposed, an alternative to the matrix Bingham distribution for modeling antipodally symmetric orientational data on $V_{k,m}$ (see Chikuse [3]).

There exists an extensive literature in the area of statistical analysis on circles, spheres, and, in general, hyperspheres, i.e., for the case $k=1$. See Watson [17], Mardia [14], and many other articles. However, there seem to remain many unsolved problems concerning distributions on general Stiefel manifolds.

In this paper, we present some distributional results on Stiefel manifolds. Section 2 gives two kinds of decomposition of the differential form for the invariant measure on $V_{k,m}$ into those for independent measures on component Stiefel manifolds and on subspaces of component rectangular matrices. A "sequential" decomposition and some applications are presented.

Section 3 is concerned with the matrix resultant or the sum $W=\sum_{i=1}^{n} X_i$ of a random sample $X_1, ..., X_n$ of size n from the $L(m, k; F)$ distribution. The singular value decomposition of W gives the maximum likelihood estimators of the population orientations and concentrations which are the corresponding component matrices in the singular value decomposition of F. We shall derive sampling distributions of matrix statistics including these sample orientations, modal orientation, and also the product matrix $W'W$. Furthermore, representations in terms of the Hankel transform and multi-sample distribution theory are briefly discussed.

Finally, it is noted that there will be an opportunity to discuss asymptotic distribution theory (Chikuse [4]), while this paper is concerned with exact expressions for distributions.

2. Decompositions of the Invariant Measure on the Stiefel Manifold and Applications

In this section, we present a decomposition of the differential form for the invariant measure on the Stiefel manifold $V_{k,m}$ into those for two independent invariant measures on component Stiefel manifolds $V_{q,m}$ and $V_{k-q,m-q}$ $(0<q<k)$. The result is a generalization of that on the orthogonal group (for $k=m$) (e.g., Muirhead [15, Lemma 9.5.3]). We then extend it to a "sequential" decomposition with more than two component Stiefel manifolds involved. The results may be useful in distribution theory and statistical inference on $V_{k,m}$. We utilize them to derive joint, marginal, and conditional distributions on component Stiefel manifolds of $V_{k,m}$ for the $L(m, k; F)$ and $B(m, k; A)$ distributions. The resulting distributions may be useful for inferential problems regarding the population distributions being concerned.

Subsequently, an alternative decomposition is briefly discussed of the differential form for the invariant measure on $V_{k,m}$ into those for independent measures on a subspace of $q \times k$ component rectangular matrices and on a component Stiefel manifold $V_{k,m-q}$ $(k \leqq q,\ k+q \leqq m)$. The result is a generalization of Herz [9, Lemma 3.7] (for $q=k$) (see also Watson [17, Eq. (2.2.2)]). Applications are illustrated.

2.1. *A Decomposition of the Invariant Measure*

Let us write $X=[V:V_1]$ with V and V_1 being $m\times q$ and $m\times(k-q)$ matrices, respectively, and let X_1 be an $m\times(m-k)$ matrix such that $H=[X\vdots X_1]\in O(m)$. We now apply the argument of Muirhead [15, Lemma 9.5.3] to the orthogonal matrix $H=[V:U]$, where $U=[V_1\vdots X_1]$. For fixed V, U can be written as $U=G(V)Y$, where $G(V)$ is any fixed matrix chosen so that $[V:G(V)]\in O(m)$, and $Y\in O(m-q)$; the relationship between U and Y is one-to-one. Writing $Y=[Z\vdots Z_1]$ with Z and Z_1 being $(m-q)\times(k-q)$ and $(m-q)\times(m-k)$ matrices, respectively, we have

$$H=[X\vdots X_1]=[V:V_1\vdots X_1]=[V:G(V)Z\vdots G(V)Z_1], \tag{2.1}$$

and, as V_1 and X_1 run over $V_{k-q,m}$ and $V_{m-k,m}$, respectively, Z and Z_1 run over $V_{k-q,m-q}$ and $V_{m-k,m-q}$, respectively; the relationships are one-to-one.

Writing $H = [\mathbf{h}_1 \cdots \mathbf{h}_q : \mathbf{h}_{q+1} \cdots \mathbf{h}_k \vdots \mathbf{h}_{k+1} \cdots \mathbf{h}_m]$, corresponding to the partition (2.1), and

$$[Z \vdots Z_1] = (\mathbf{g}_1 \cdots \mathbf{g}_{k-q} \vdots \mathbf{g}_{k-q+1} \cdots \mathbf{g}_{m-q}],$$

we have

$$\mathbf{h}_{q+j} = G(V)\,\mathbf{g}_j, \qquad j = 1, ..., m-q, \tag{2.2}$$

and hence

$$d\mathbf{h}_{q+j} = G(V)\,d\mathbf{g}_j, \qquad j = 1, ..., m-q. \tag{2.3}$$

Now we rewrite the differential form (1.1) as

$$(X'\,dX) = \left[\bigwedge_{i<j} {}_1^q\, \mathbf{h}_j'\, d\mathbf{h}_i \bigwedge_{j=1}^{m-q} \bigwedge_{i=1}^{q} \mathbf{h}_{q+j}'\, d\mathbf{h}_i \right]$$
$$\times \left[\bigwedge_{i<j} {}_1^{k-q}\, \mathbf{h}_{q+j}'\, d\mathbf{h}_{q+i} \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k-q} \mathbf{h}_{k+j}'\, d\mathbf{h}_{q+i} \right]. \tag{2.4}$$

The first term of the right-hand side of (2.4) is seen to be the differential form $(V'\,dV)$ for the invariant measure on $V_{q,m}$, and the second term becomes, with (2.2) and (2.3) being substituted,

$$\bigwedge_{i<j} {}_1^{k-q}\, \mathbf{g}_j'\, d\mathbf{g}_i \bigwedge_{j=1}^{m-k} \bigwedge_{i=1}^{k-q} \mathbf{g}_{k-q+j}'\, d\mathbf{g}_i = (Z'\,dZ),$$

which is the differential form for the invariant measure on $V_{k-q,m-q}$. Dividing by the volumes of the invariant measures establishes

THEOREM 2.1. *Let us write a random matrix X on $V_{k,m}$ as $X = [V : V_1]$ with V and V_1 being $m \times q$ and $m \times (k-q)$ matrices, respectively $(0 < q < k)$. Then we can write $V_1 = G(V)Z$, where $G(V)$ is any matrix chosen so that $[V \vdots G(V)]$ is orthogonal, and as V_1 runs over $V_{k-q,m}$, Z runs over $V_{k-q,m-q}$, and the relationship is one-to-one. The differential form $[dX]$ for the normalized invariant measure on $V_{k,m}$ is decomposed as the product*

$$[dX] = [dV][dZ] \tag{2.5}$$

of those $[dV]$ and $[dZ]$ on $V_{q,m}$ and $V_{k-q,m-q}$, respectively.

The referee suggests the following proof of the decomposition (2.5) by a group theoretical approach, alternative to the above proof in terms of differential forms. Using the identification $V_{k,m} = O(m)/O(m-k)$ as the quotient space (see, e.g., Farrell [8, p. 119]), we can show that the trivial function from $V_{k,m}$ onto $V_{q,m}$ induces the identification $V_{k,m} = V_{q,m} \times V_{k-q,m-q}$ as the Cartesian product, which is also indicated by (2.1) in the above proof. Then, the decomposition (2.5) follows immediately from the fact that the uniform distribution on $V_{k,m}$ (etc.) is the unique probability distribution on $V_{k,m}$ which is invariant under the actions of $O(m)$ (see Farrell [8, Theorem 3.4.1]).

An Extension

Next, we give a "sequential" decomposition. We write $X_1 = X$, $m_1 = m$, and $k_1 = k$ for the notational convenience in the following. Writing $X_1 = [X_{11} : X_{12}] \in V_{k_1,m_1}$, where X_{11} and X_{12} are $m_1 \times q_1$ and $m_1 \times (k_1 - q_1)$ matrices, respectively $(0 < q_1 < k_1)$, we have that $X_{12} = G_1 X_2$, where $X_2 \in V_{k_1 - q_1, m_1 - q_1}$ and $G_1 = G_1(X_{11})$ is chosen so that $[X_{11} \vdots G_1] \in O(m_1 - q_1)$, and that

$$[dX_1] = [dX_{11}][dX_2],$$

using Theorem 2.1.

Next, we apply the argument to X_2 and put $m_2 = m_1 - q_1$ and $k_2 = k_1 - q_1$. Writing $X_2 = [X_{21} : X_{22}] \in V_{k_2,m_2}$, where X_{21} and X_{22} are $m_2 \times q_2$ and $m_2 \times (k_2 - q_2)$ matrices, respectively $(0 < q_2 < k_2)$, we have that $X_{22} = G_2 X_3$, where $X_3 \in V_{k_2 - q_2, m_2 - q_2}$ and $G_2 = G_2(X_{21})$ is chosen so that $[X_{21} \vdots G_2] \in O(m_2 - q_2)$, and that

$$[dX_2] = [dX_{21}][dX_3].$$

Thus, continuing this way, we may establish

COROLLARY 2.1. *For the random matrix X_1 on V_{k_1,m_1}, we can write "sequentially" $X_i = [X_{i1} : G_i X_{i+1}] \in V_{k_i,m_i}$, where $X_{i1} \in V_{q_i,m_i}$, $X_{i+1} \in V_{k_i - q_i, m_i - q_i}$, $G_i = G_i(X_{i1})$ is chosen so that $[X_{i1} \vdots G_i] \in O(m_i - q_i)$, and we put $m_i = m_{i-1} - q_{i-1}$ and $k_i = k_{i-1} - q_{i-1}$ $(0 < q_{i-1} < k_{i-1})$, for $i = 1, 2, \ldots$. The differential form $[dX_1]$ for the normalized invariant measure on V_{k_1,m_1} is decomposed as the product*

$$[dX_1] = [dX_{11}][dX_{21}] \cdots [dX_{l1}][dX_{l+1}], \tag{2.6}$$

of those on the respective component Stiefel manifolds $V_{q_1,m_1}, V_{q_2,m_2}, \ldots, V_{q_l,m_l}$, and $V_{k_l - q_l, m_l - q_l}$, for $l = 1, 2, \ldots$.

2.2. *Applications*

It is seen from Corollary 2.1 that the density $f_1(X_1)$ of an $m_1 \times k_1$ random matrix X_1 on V_{k_1,m_1} is expressed as the product

$$f_1(X_1) = f_{11}(X_{11}) f_{21}(X_{21} \mid X_{11}) \cdots f_{l1}(X_{l1} \mid X_{11}, ..., X_{l-1,1}) \times f_{l+1}(X_{l+1} \mid X_{11}, ..., X_{l1}), \tag{2.7}$$

of the marginal density of X_{11} on V_{q_1,m_1}, the conditional density of X_{21} on V_{q_2,m_2} given $X_{11}, ...,$ the conditional density of X_{l1} on V_{q_l,m_l} given $X_{11}, ..., X_{l-1,1}$, and the conditional density of X_{l+1} on $V_{k_l-q_l,m_l-q_l}$ given $X_{11}, ..., X_{l1}$, for $l = 1, 2, ...$.

(i) *Matrix Langevin Distribution*

Suppose that X_1 has the $L(m_1, k_1; F_1)$ distribution. Writing $F_1 = [F_{11} : F_{12}]$, corresponding to $X_1 = [X_{11} : G_1 X_2]$ $(G_1 = G_1(X_{11}))$, it is easily shown from (2.7) for $l = 1$ that

$$f_{11}(X_{11}) = [{}_0F_1^{(k_1)}(\tfrac{1}{2}m_1; \tfrac{1}{4}F_1'F_1)]^{-1}\, etr(F_{11}'X_{11}) \times {}_0F_1^{(k_1-q_1)}(\tfrac{1}{2}(m_1-q_1); \tfrac{1}{4}F_{12}'G_1G_1'F_{12}), \tag{2.8}$$

and

$$f_2(X_2 \mid X_{11}) = [{}_0F_1^{(k_1-q_1)}(\tfrac{1}{2}(m_1-q_1); \tfrac{1}{4}F_{12}'G_1G_1'F_{12})]^{-1}\, etr(F_{12}'G_1X_2). \tag{2.9}$$

In general, writing

$$\underbrace{F_{12...2}}_{l} = [\underbrace{F_{12...21}}_{l+1} : \underbrace{F_{12...2}}_{l+1}], \qquad \text{corresponding to} \quad X_l = [X_{l1} : G_l X_{l+1}]$$

$(G_l = G_l(X_{l1}))$, we have

$$\begin{aligned} f_{l1}(X_{l1} \mid X_{11}, ..., X_{l-1,1}) &= [{}_0F_1^{(k_l)}(\tfrac{1}{2}m_l; \tfrac{1}{4}F_{12...2}'G_1 \cdots G_{l-1} \\ &\quad \times G_{l-1}' \cdots G_1'F_{12...2})]^{-1}\, etr(F_{12...21}'G_1 \cdots G_{l-1}X_{l1}) \\ &\quad \times {}_0F_1^{(k_l-q_l)}(\tfrac{1}{2}(m_l-q_l); \tfrac{1}{4}F_{12...2}'G_1 \cdots G_l G_l' \cdots G_1'F_{12...2}), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} f_{l+1}(X_{l+1} \mid X_{11}, ..., X_{l1}) &= [{}_0F_1^{(k_l-q_l)}(\tfrac{1}{2}(m_l-q_l); \tfrac{1}{4}F_{12...2}'G_1 \cdots G_l \\ &\quad \times G_l' \cdots G_1'F_{12...2})]^{-1}\, etr(F_{12...2}'G_1 \cdots G_lX_{l+1}), \\ &\quad \text{for} \quad l = 1, 2, ... \end{aligned} \tag{2.11}$$

Thus, we obtain the density (2.7), with (2.8)–(2.11) being substituted, for the $L(m_1, k_1; F_1)$ distribution.

It is noted that the densities $f_{l1}(X_{l1} \mid X_{11}, ..., X_{l-1,1})$, $l = 1, 2, ...$ (with $X_{01} = \phi$, the null set), are of similar forms which are *not of Langevin type*, while the density $f_{l+1}(X_{l+1} \mid X_{11}, ..., X_{l1})$ is the matrix Langevin $L(k_l - q_l, m_l - q_l; G_l' \cdots G_1' F_{12 \ldots 2})$, $l = 1, 2, \ldots$. Khatri and Mardia [13] considered the case for $l = 1$ and derived the marginal density (2.8) using a different method. They obtained the distribution of $X_{12} = G_1 X_2$ given X_{11} in the form of a *degenerate* matrix Langevin distribution.

It is seen from (2.8) and (2.9) that, for the case $F_{12} = 0$, X_{11} and X_2 are independent, X_{11} has the $L(m_1, q_1; F_{11})$ distribution, and X_2 is uniformly distributed on $V_{k_1 - q_1, m_1 - q_1}$. In general, for the case $F_{12\ldots2} = 0$, $\{X_{11}, ..., X_{l1}\}$ and X_{l+1} are independent, X_{l1} given $\{X_{11}, ..., X_{l-1,1}\}$ has the $L(m_l, q_l; G_{l-1}' \cdots G_1' F_{12 \ldots 21})$ distribution, and X_{l+1} is uniformly distributed on $V_{k_l - q_l, m_l - q_l}$ for $l = 1, 2, \ldots$.

(ii) *Matrix Bingham Distribution*

Suppose that X_1 has the $B(m_1, k_1; A)$ distribution. It is easily shown from (2.7) for $l = 1$ that

$$\begin{aligned} f_{11}(X_{11}) = {} & [{}_1F_1^{(k_1)}(\tfrac{1}{2}k_1; \tfrac{1}{2}m_1; A)]^{-1} \, etr(X_{11}' A X_{11}) \\ & \times {}_1F_1^{(k_1 - q_1)}(\tfrac{1}{2}(k_1 - q_1); \tfrac{1}{2}(m_1 - q_1); G_1' A G_1) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} f_2(X_2 \mid X_{11}) = {} & [{}_1F_1^{(k_1 - q_1)}(\tfrac{1}{2}(k_1 - q_1); \tfrac{1}{2}(m_1 - q_1); G_1' A G_1)]^{-1} \\ & \times etr(X_2' G_1' A G_1 X_2). \end{aligned} \tag{2.13}$$

In general, we have

$$\begin{aligned} & f_{l1}(X_{l1} \mid X_{11}, ..., X_{l-1,1}) \\ & \quad = [{}_1F_1^{(k_l)}(\tfrac{1}{2}k_l; \tfrac{1}{2}m_l; G_{l-1}' \cdots G_1' A G_1 \cdots G_{l-1})]^{-1} \\ & \qquad \times etr(X_{l1}' G_{l-1}' \cdots G_1' A G_1 \cdots G_{l-1} X_{l1}) \\ & \qquad \times {}_1F_1^{(k_l - q_l)}(\tfrac{1}{2}(k_l - q_l); \tfrac{1}{2}(m_l - q_l); G_l' \cdots G_1' A G_1 \cdots G_l) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} & f_{l+1}(X_{l+1} \mid X_{11}, ..., X_{l1}) \\ & \quad = [{}_1F_1^{(k_l - q_l)}(\tfrac{1}{2}(k_l - q_l); \tfrac{1}{2}(m_l - q_l); G_l' \cdots G_1' A G_1 \cdots G_l)] \\ & \qquad \times etr(X_{l+1}' G_l' \cdots G_1' A G_1 \cdots G_l X_{l+1}). \end{aligned} \tag{2.15}$$

Thus, we obtain the density (2.7), with (2.12)–(2.15) being substituted, for the $B(m_1, k_1; A)$ distribution.

An observation similar to that for the previous case (i) is noted that the densities $f_{l1}(X_{l1} \mid X_{11}, ..., X_{l-1,1})$, $l = 1, 2, ...$ (with $X_{01} = \phi$), are of similar forms which are *not of Bingham type,* while the density $f_{l+1}(X_{l+1} \mid X_{11}, ..., X_{l1})$ is the matrix Bingham $B(k_l - q_l, m_l - q_l; G'_l \cdots G'_1 A G_1 \cdots G_l)$, $l = 1, 2, ...$.

2.3. *An Alternative Decomposition of the Invariant Measure*

THEOREM 2.2. *Let us write a random matrix X on $V_{k,m}$ as*

$$X = [X'_1 : (I_k - X'_1 X_1)^{1/2}\, U']'. \tag{2.16}$$

where X_1 is a $q \times k$ matrix such that $I_k - X'_1 X_1$ is positive definite and U is an $(m-q) \times k$ matrix in $V_{k,m-q}$ $(k \leqq q,\ k+q \leqq m)$. Then, the differential form $[dX]$ for the normalized invariant measure on $V_{k,m}$ is decomposed as

$$[dX] = c_0 |I_k - X'_1 X_1|^{(m-q-k-1)/2}\, [dU]\, (dX_1), \tag{2.17}$$

in terms of $[dU]$ on $V_{k,m-q}$ and (dX_1), where

$$c_0 = \Gamma_k(m/2)/\pi^{kq/2} \Gamma_k((m-q)/2). \tag{2.18}$$

Proof. The proof is essentially due to Herz [9, Lemma 3.7] for $q = k$. It is carried out by utilizing the uniqueness of Laplace transforms of two measures, the Jacobian of the transformation $X_1 \to X'_1 X_1$, the convolution theorem for the Laplace transforms and some formulae for the hypergeometric functions of matrix argument.

For the case $k = 1$, Watson [17, Eq. (2.2.1) and (2.2.2)] expressed a random vector on $V_{1,m}$ as a direct sum of two orthogonal components and gave a decomposition of the invariant measure on $V_{1,m}$ into two independent measures. Theorem 2.2 is a multivariate generalization of his result.

Applications

Suppose that X has the $L(m, k; F)$ distribution, where $F = [F'_1 : F'_2]'$ according to the partition (2.16). Then, it is readily shown that U given X_1 has the $L(m-q, k; F_2(I - X'_1 X_1)^{1/2})$ distribution and then that the density of X_1 is

$$\begin{aligned} f_{X_1}(X_1) = {} & c_0 [{}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}F'F)]^{-1}\, etr(F'_1 X_1)\, |I - X'_1 X_1|^{(m-q-k-1)/2} \\ & \times {}_0F_1^{(k)}(\tfrac{1}{2}(m-q); \tfrac{1}{4}F_2(I_k - X'_1 X_1)\, F'_2), \end{aligned} \tag{2.19}$$

where c_0 is given by (2.18).

Hence, if $F_2=0$, U and X_1 are independent, U is uniformly distributed on $V_{k,m-q}$, and X_1 has the density

$$f_{X_1}^{(0)}(X_1)=c_0[{}_0F_1^{(k)}(\tfrac{1}{2}m;\tfrac{1}{4}F_1'F_1)]^{-1}\,etr(F_1'X_1)\,|I-X_1'X_1|^{(m-q-k-1)/2}. \tag{2.20}$$

If furthermore $F_1=0$, the density of X_1 becomes $c_0|I-X_1'X_1|^{(m-q-k-1)/2}$ (Beta-type).

Next, for the case when X has the $B(m,k;A)$ distribution, where

$$A=\begin{array}{c} \begin{array}{cc} q & m-q \end{array} \\ \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{12}' & A_{22} \end{array}\right] \end{array}\begin{array}{l} \\ q \\ m-q. \end{array}$$

We just note that U given X_1 has the *generalized* matrix Bingham distribution (in the sense of Khatri and Mardia [13]) with the density proportional to $etr[U(I_k-X_1'X_1)\,U'A_{22}+2(I_k-X_1'X_1)^{1/2}\,X_1'A_{12}U]$.

3. Sampling Distributions in the Matrix Langevin Population Distributions

3.1. *Matrix Langevin Distribution*

We assume that the parameter matrix F in the matrix Langevin $L(m,k;F)$ distribution with the density (1.3) is of full rank k, since there is no loss of essential points in te discussion of this section. We write the singular value decomposition of F as

$$F=\Gamma\Delta\Theta', \tag{3.1}$$

where $\Gamma\in V_{k,m}$ with the first nonzero element in each column positive, $\Theta\in O(k)$, and $\Delta=\mathrm{diag}(\lambda_1,\ldots,\lambda_k)$, $\lambda_1\geqq\cdots\geqq\lambda_k>0$; the decomposition (3.1) is unique if $\lambda_1>\cdots>\lambda_k>0$. The matrix parameters in (3.1) may have the following meanings extended from those for the case $k=1$. Γ and Θ are "orientations," while the diagonal elements of Δ are "concentration" parameters in the k directions determined by Γ and Θ. The $L(m,k;F)$ distribution is a uni-modal distribution with the "population modal orientation" $M=\Gamma\Theta'$, and is "rotationally symmetric" around M; i.e., the density takes its maximum value at $X=M$, and the density is unchanged under the simultaneous transformations $M\to Q_1MQ_2'$ and $X\to Q_1XQ_2'$, for $Q_1\in O(m)$ and $Q_2\in O(k)$.

Let $X_1, ..., X_n$ be a random sample of size n from the $L(m, k; F)$ distribution, and let $W = \sum_{i=1}^n X_i$ be its matrix resultant or sum. Let us write the "unique" singular value decomposition of W as

$$W = H_1 W_d H_2', \tag{3.2}$$

where $H_1 \in V_{k,m}$ with the elements of the first row of H_1 positive, $H_2 \in O(k)$, and $W_d = \mathrm{diag}(w_1, ..., w_k)$, $w_1 > \cdots > w_k > 0$. We can also consider another "unique" (polar) decomposition of W,

$$W = H_W T_W^{1/2}, \qquad \text{with} \quad H_W = W(W'W)^{-1/2} \text{ and } T_W = W'W, \tag{3.3}$$

i.e., $H_W = H_1 H_2'$ and $T_W^{1/2} = H_2 W_d H_2'$ is the unique (positive definite) square root of $T_W = H_2 W_d^2 H_2'$. It is noted that the "uniqueness" of each of the decompositions (3.2) and (3.3) is ensured by the fact that the statistic W is of full rank "almost everywhere." It has been shown (e.g., Khatri and Mardia [13], Jupp and Mardia [12]) that H_1 and H_2 are the maximum likelihood estimators of the orientation parameters Γ and Θ, respectively. $H_W = H_1 H_2'$ is the maximum likelihood estimator of $M = \Gamma\Theta'$ (Downs [7]) and may be called the "sample modal orientation." $T_W = W'W$ indicates the inner products of the columns of W and may be of importance in testing problems. H_W and T_W become the mean direction and the length, respectively, of the resultant of a random sample for $k = 1$.

In this section, we derive sampling distributions of $\{H_1, H_2, W_d\}$ and $\{H_W, T_W\}$. Subsequently, the expression in terms of the Hankel transform for the distribution of T_W and multi-sample distributions are considered.

Since the matrix Langevin distributions form an exponential family, we can write the density of W as (see Khatri and Mardia [13, Eq. (3.5)])

$$f_W(W; F) = a_m(F)^{-n}\, etr(F'W)\, f_W(W; 0), \tag{3.4}$$

where $f_W(W; 0)$ is the density of W when $\Delta = 0$ (i.e., $F = 0$) and is obtained by inverting the Laplace transform as

$$f_W(W; 0) = [(2\pi^{1/2})^{km}\, \Gamma_k(m/2)]^{-1} \int_{R>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}WRW') \times [{}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}R)]^n\, |R|^{(m-k-1)/2}\, (dR), \tag{3.5}$$

with the integration over the space of $k \times k$ positive definite matrices, and, for the rest of Section 3, we use the notation

$$a_m(F) = {}_0F_1^{(k)}(m/2; F'F/4). \tag{3.6}$$

3.2. *Distributions of H_1, H_2, and W_d*

In the sequel, we let H_1 run over the entire $V_{k,m}$ so that the volume $\int_{V_{k,m}}(H_1' \, dH_1) = 2^{-k}w(k, m)$, where $w(k, m)$ is given by (1.2). Utilizing the decomposition of (dW) as the product of $[dH_1]$, $[dH_2]$ and $\prod_{i=1}^{k} dw_i$ (e.g., James [10, Eq. (8.8)]), we readily obtain the joint density of H_1, H_2, and W_d, i.e., $w_1, ..., w_k$,

$$f_{H_1,H_2,W_d}(H_1, H_2, W_d; F) = a_m(F)^{-n} \, etr(F'H_1 W_d' H_2') \, f_{W_d}(W_d; 0), \quad (3.7)$$

where

$$\begin{aligned} f_{W_d}(W_d; 0) = \{&\pi^{k^2/2}/2^{k(m-1)} \Gamma_k(k/2)[\Gamma_k(m/2)]^2\} \\ &\times \prod_{i=1}^{k} w_i^{m-k} \prod_{i<j}{}^{k}_{1} (w_i^2 - w_j^2) \int_{R>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; W_d^2, -\tfrac{1}{4}R) \\ &\times [{}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}R)]^n \, |R|^{(m-k-1)/2} \, (dR) \qquad (3.8) \end{aligned}$$

is the density of W_d when $\Delta = 0$, where ${}_0F_1^{(k)}(m/2; W_d^2, -R/4)$ is the hypergeometric function of two matrix arguments.

It is seen from (3.7) that, when $\Delta = 0$, i.e., the population distribution is uniform on $V_{k,m}$, H_1, H_2, and W_d are mutually independent, H_1 and H_2 are uniformly distributed on $V_{k,m}$ and $O(k)$, respectively, and W_d has the density (3.8).

Integrating over $H_2 \in O(k)$ and then over $H_1 \in V_{k,m}$ in (3.7) gives the joint density of H_1 and W_d,

$$f_{H_1,W_d}(H_1, W_d; F) = a_m(F)^{-n} \, a_k(F'H_1 W_d) \, f_{W_d}(W_d; 0), \quad (3.9)$$

and the density of W_d,

$$f_{W_d}(W_d; F) = a_m(F)^{-n} \, {}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}\Delta^2, W_d^2) \, f_{W_d}(W_d; 0), \quad (3.10)$$

where $f_{W_d}(W_d; 0)$ is given by (3.8).

Dividing (3.7) by (3.10) gives the conditional joint density of (H_1, H_2) given W_d:

$$f_{H_1,H_2|W_d}(H_1, H_2 \mid W_d; F) = [{}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}\Delta^2, W_d^2)]^{-1} \, etr(F'H_1 W_d H_2'). \quad (3.11)$$

The distribution given by (3.11) may be called the matrix "jointly Langevin" distribution with multiple parameters F and W_d. Dividing (3.9) by (3.10) gives the conditional density of H_1 given W_d

$$f_{H_1|W_d}(H_1 \mid W_d; F) = [{}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}\Delta^2, W_d^2)]^{-1} \, a_k(F'H_1 W_d), \quad (3.12)$$

which is apparently *not of Langevin type*. Dividing (3.7) by (3.9) gives the conditional density of H_2 given (H_1, W_d),

$$f_{H_2 \mid H_1, W_d}(H_2 \mid H_1, W_d; F) = a_k(F'H_1 W_d)^{-1} \operatorname{etr}(W_d H_1' F \cdot H_2), \quad (3.13)$$

which is the density of the $L(k, k; F'H_1 W_d)$ distribution. Similarly, we obtain the conditional density of H_2 given W_d, (3.12) with $a_k(F'H_1 W_d)$ replaced by $a_m(FH_2 W_d)$, which is *not of Langevin type*, and the conditional $L(m, k; FH_2 W_d)$ distribution of H_1 given (H_2, W_d).

It may be worth noting that many of the above results on joint, marginal, and conditional distributions are simple consequences of the fact that the matrix Langevin distributions form an exponential family and are rotationally symmetric.

3.3. *Distributions of H_W and T_W*

Our derivation of the distributions of H_w and T_w is based on the following lemma, which is essentially due to Herz [9, Lemma 1.4] (see also James [10, identity (8.19)] and Muirhead [15, Theorem 2.1.14]) and will be frequently used for the rest of this section.

LEMMA 3.1. *Let Z be an $m \times k$ random matrix and write the polar decomposition of Z as*

$$Z = H_Z T_Z^{1/2}, \qquad \text{with} \quad H_Z = Z(Z'Z)^{-1/2} \qquad \text{and} \qquad T_Z = Z'Z.$$

Then, we have

$$(dZ) = [\pi^{km/2}/\Gamma_k(m/2)] \, |T_Z|^{(m-k-1)/2} \, [dH_Z] \, (dT_Z). \quad (3.14)$$

Using this lemma with (3.4), the joint density of H_W and T_W is

$$f_{H_W, T_W}(H_W, T_W; F) = a_m(F)^{-n} \operatorname{etr}(F'H_W T_W^{1/2}) f_{T_W}(T_W; 0), \quad (3.15)$$

where

$$f_{T_W}(T_W; 0) = c|T_W|^{(m-k-1)/2} \int_{R>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}T_W R) \times [{}_0F_1(\tfrac{1}{2}m; -\tfrac{1}{4}R)]^n \, |R|^{(m-k-1)/2}(dR), \quad (3.16)$$

is the density of T_W when $\Delta = 0$, with

$$c = \{2^{km}[\Gamma_k(m/2)]^2\}^{-1}. \quad (3.17)$$

It is seen from (3.15) that when the population distribution is uniform on

$V_{k,m}$, H_W and T_W are independent, H_W is uniformly distributed (these two facts have been noticed by, e.g., Downs [7]), and T_W has the density (3.16).

Integrating over H_W and T_W yield, respectively, the density of T_W,

$$f_{T_W}(T_W; F) = a_m(F)^{-n} \, {}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}FT_W F') \, f_{T_W}(T_W; 0), \tag{3.18}$$

and the density of H_W,

$$f_{H_W}(H_W; F) = a_m(F)^{-n} \int_{T_W > 0} etr(F' H_W T_W^{1/2}) \, f_{T_W}(T_W; 0)(dT_W), \tag{3.19}$$

which is *not of Langevin type*. It is noted that (3.18) has been obtained by a different method in Khatri and Mardia [13, Eq. (3.6)]. Dividing (3.15) by (3.18) yields the conditional density of H_W given T_W, which is the density of the $L(m, k; FT_W^{1/2})$ distribution (this fact has been already derived by Downs [7]).

Expression as Hankel Transform

Mardia [14, Section 4.2] showed that the distribution of $\mathbf{X}'\mathbf{X}$ for an arbitrary random vector $\mathbf{X}$ (hence for $k = 1$) is expressed in terms of the Hankel transform in scalar argument. We shall see how his result is extended to our general case $k \geqq 1$. Herz [9, (3.1)] defined the Hankel transform in a $k \times k$ matrix argument as

$$g(T) = \int_{R > 0} A_\gamma(TR) \, |R|^\gamma f(R)(dR), \tag{3.20}$$

which is denoted by $g(T) = [U_\gamma f](T)$, where $A_\gamma(R)$ $(= {}_0F_1^{(k)}(\gamma + (k+1)/2; -R)/\Gamma_k(\gamma + (k+1)/2))$ is the Herz's Bessel function.

Let Z be an arbitrary $m \times k$ random matrix with the characteristic function $\Psi(S)$ for an $m \times k$ matrix S, and suppose that $\Psi(S)$ is integrable. We shall now obtain the density of T_Z, where $Z = H_Z T_Z^{1/2}$, with $H_Z = Z(Z'Z)^{-1/2}$ and $T_Z = Z'Z$. Let us write $S = H_S T_S^{1/2}$, with $H_S = S(S'S)^{-1/2}$ and $T_S = S'S$, and put $\Psi(S) = \bar{\Psi}(H_S, T_S)$.

Using the inversion theorem and Lemma 3.1 leads to the joint density of H_Z and T_Z,

$$f_{H_Z, T_Z}(H_Z, T_Z) = c|T_Z|^{(m-k-1)/2} \int_{T_S > 0} \int_{V_{k,m}} etr(-IT_Z^{1/2} H_Z' H_S T_S^{1/2}) \times \bar{\Psi}(H_S, T_S) \, |T_S|^{(m-k-1)/2} \, [dH_S] \, (dT_S), \tag{3.21}$$

where c is given by (3.17). Integrating over $H_Z \in V_{k,m}$ in (3.21) gives the density of T_Z,

$$f_{T_Z}(T_Z) = c|T_Z|^{(m-k-1)/2} \int_{T_S>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}T_Z T_S) \times \bar{\Psi}_1(T_S)\,|T_S|^{(m-k-1)/2}\,(dT_S), \tag{3.22}$$

where we put

$$\bar{\Psi}_1(T_S) = \int_{V_{k,m}} \bar{\Psi}(H_S, T_S)\,[dH_S]. \tag{3.23}$$

Putting $T_S/4 = R$, (3.22) can be rewritten as

$$f_{T_Z}(T_Z) = [\Gamma_k(m/2)]^{-1}\,|T_Z|^{(m-k-1)/2}\,[U_\gamma f]\,(T_Z), \tag{3.24}$$

in terms of the Hankel transform (3.20) with $\gamma = (m-k-1)/2$ and $f(R) = \bar{\Psi}_1(4R)$.

Next, we shall express $\bar{\Psi}_1(4R)$ in the expression (3.24) in terms of $f_{T_Z}(T_Z)$. From the definition we can write

$$\bar{\Psi}_1(4R) = \int_{V_{k,m}} \left\{ \int_{T_Z>0} \int_{V_{k,m}} etr(2iT_Z^{1/2} H'_Z H_S R^{1/2}) \times f_{H_Z,T_Z}(H_Z, T_Z)\,[dH_Z]\,(dT_Z) \right\} [dH_S], \tag{3.25}$$

where $f_{H_Z,T_Z}(H_Z, T_Z)$ is the joint density of H_Z and T_Z. Integrating first over $H_S \in V_{k,m}$ and then over $H_Z \in V_{k,m}$ in (3.25) yields

$$\begin{aligned} \bar{\Psi}_1(4R) &= \int_{T_Z>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; -RT_Z)\,f_{T_Z}(T_Z)\,(dT_Z) \\ &= \Gamma_k(m/2)[U_\gamma g]\,(R), \end{aligned} \tag{3.26}$$

where $\gamma = (m-k-1)/2$ and $g(T_Z) = |T_Z|^{(-m+k+1)/2} f_{T_Z}(T_Z)$, which is the inverse relationship of (3.24).

We now assume that $f_{T_Z}(T_Z)$ is invariant under the transformation $T_Z \to QT_ZQ'$, $Q \in O(k)$; this assumption is satisfied by our $f_{T_W}(T_W; 0)$. Making the transformation $T_Z \to QT_ZQ'$, $Q \in O(k)$, and then integrating over $Q \in O(k)$ in (3.26) gives

$$\begin{aligned} \bar{\Psi}_1(-4R) &= \int_{T_Z>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; R, T_Z)\,f_{T_Z}(T_Z)\,(dT_Z) \\ &= \sum_{l=0}^{\infty} \sum_{\lambda} \left[\int_{T_Z>0} C_\lambda(T_Z)\,f_{T_Z}(T_Z)\,(dT_Z) \right] \\ &\quad \times C_\lambda(R)/l!\,(m/2)_\lambda\,C_\lambda(I_k). \end{aligned} \tag{3.27}$$

Then each integral on the last line in (3.27) may be a "moment of T_Z indexed by λ," $E[C_\lambda(T_Z)]$, and hence $\bar{\Psi}_1(-4R)$ may be considered as a moment generating function of T_Z. Thus, the moments of symmetric functions and monomials of latent roots of T_Z are obtained.

Now, we are concerned with $Z = W = \sum_{i=1}^n X_i$, where $X_1, ..., X_n$ is a random sample of size n from the $L(m, k; F)$ distribution. When $F = 0$ (i.e., $\Delta = 0$), we have $\bar{\Psi}(H_S, T_S) = [{}_0F_1^{(k)}(\frac{1}{2}m; -\frac{1}{4}T_S)]^n$; hence

$$f(R) = [{}_0F_1^{(k)}(\tfrac{1}{2}m; -R)]^n, \tag{3.28}$$

and (3.24), with $Z = W$ and $f(R)$ given by (3.28), gives the density of T_W, which is equivalent to (3.16). Substituting this result into (3.18) yields the density of T_W for general $F \neq 0$ in terms of the Hankel transform.

Expanding the moment generating function $[{}_0F_1^{(k)}(\frac{1}{2}m; R)]^n$ in terms of the zonal polynomials $C_\lambda(R)$ and then comparing with the coefficients of the $C_\lambda(R)$ in (3.27) yields the moments $E[C_\lambda(T_Z)]$ for $F = 0$. They are, e.g.,

$$\begin{aligned} E[C_{(1)}(T_Z)]/C_{(1)}(I_k) &= n \qquad (\text{trivial since } C_{(1)}(T_Z) = \operatorname{tr}(T_Z)) \\ E[C_{(2)}(T_Z)]/C_{(2)}(I_k) &= nm^{-1}[m + (m+2)(n-1)\, g_{(1),(1)}^{(2)}], \\ E[C_{(1,1)}(T_Z)]/C_{(1,1)}(I_k) &= nm^{-1}[m + (m-1)(n-1)\, g_{(1),(1)}^{(1,1)}], \end{aligned} \tag{3.29}$$

where the coefficient g's are defined in terms of the invariant polynomials with two matrix arguments (e.g., Davis [6, Eq. (2.10)]).

Multi-sample Case

Let X_{jl}, $l = 1, ..., n_j$, $j = 1, ..., q$, be independent random samples of sizes $n_1, ..., n_q$ from the $L(m, k; F_j)$ distributions, $j = 1, ..., q$, respectively. Let us write

$$\begin{aligned} W_j = \sum_{l=1}^{n_j} X_{jl} = H_{W_j} T_{W_j}^{1/2}, &\qquad \text{with} \quad H_{W_j} = W_j (W_j' W_j)^{-1/2} \\ \text{and} \qquad T_{W_j} = W_j' W_j, &\qquad j = 1, ..., q, \\ W = \sum_{j=1}^{q} W_j = H_W T_W^{1/2}, &\qquad \text{with} \quad H_W = W(W'W)^{-1/2} \\ \text{and} \qquad T_W = W'W, & \end{aligned}$$

and $T^* = (T_{W_1}, ..., T_{W_q})$.

We can obtain the densities of T_W, T_W given T^*, and T^* by a multivariate generalization of the algebra in Mardia [14, Section 4.6] for $k = 1$, using Lemma 3.1 repeatedly. Then, we are lead to the conditional density of T^*, given T_W,

$$f_{T^*|T_W}(T^* \mid T_W)$$
$$= \prod_{j=1}^{q} f_{T_{W_j}}(T_{W_j}; 0) \int_{T_S>0} \int_{V_{k,m}} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}T_W T_S)$$
$$\times \prod_{j=1}^{q} [{}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}T_{W_j}(F_j + iH_S T_S^{1/2})' (F_j + iH_S T_S^{1/2}))]^{n_j}$$
$$\times |T_S|^{(m-k-1)/2} [dH_S] (dT_S) \Big/ \int_{T_S>0} \int_{V_{k,m}} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}T_W T_S)$$
$$\times \prod_{j=1}^{q} [{}_0F_1^{(k)}(\tfrac{1}{2}m; \tfrac{1}{4}(F_j + iH_S T_S^{1/2})' (F_j + iH_S T_S^{1/2}))]^{n_j}$$
$$\times |T_S|^{(m-k-1)/2} [dH_S] (dT_S), \tag{3.30}$$

where $f_{T_{W_j}}(T_{W_j}; 0)$ is the density of T_{W_j} when $F_j = 0, j = 1, ..., q$.

When $F_1 = \cdots = F_q = F$ (the case of homogeneity), (3.30) is simplified as

$$f_{T^*|T_W}(T^* \mid T_W)$$
$$= \prod_{j=1}^{q} f_{T_{W_j}}(T_{W_j}; 0) \int_{R>0} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}T_W R)$$
$$\times \prod_{j=1}^{q} {}_0F_1^{(k)}(\tfrac{1}{2}m; -\tfrac{1}{4}T_{W_j} R) |R|^{(m-k-1)/2} (dR) / f_{T_W}(T_W; 0), \tag{3.31}$$

which *does not depend on F.*

Before closing this paper, it is noted that, for the antipodally symmetric Bingham $B(m, k; A)$ distribution, the statistic $S = \sum_{i=1}^{n} X_i X_i'$ is of more interest, where $X_1, ..., X_n$ is a random sample of size n from the $B(m, k; A)$ distribution, and various distributional results, including those related to S, are readily obtained, analogously to those for $k = 1$ due to Bingham [1].

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